WELCOME

07-July -2023

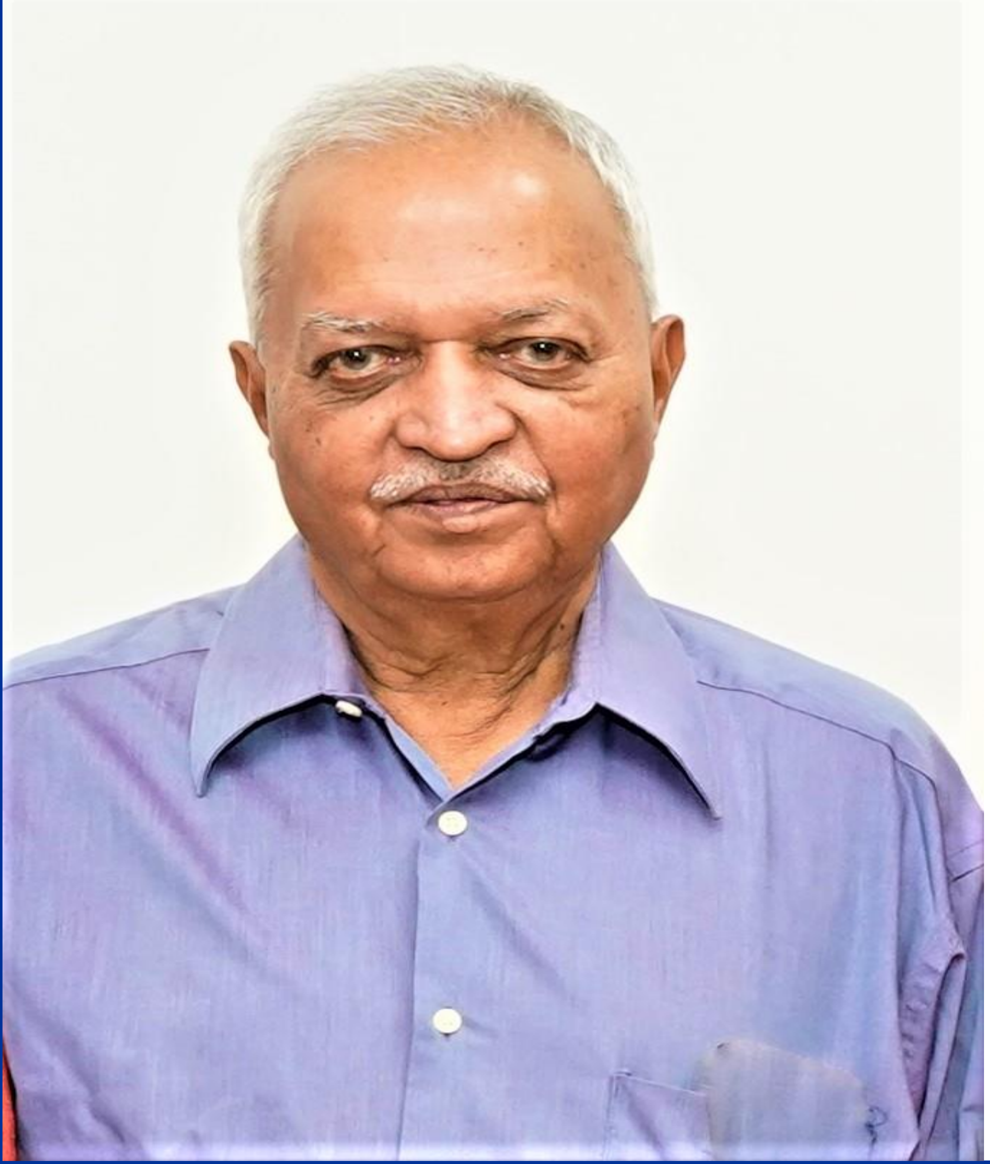
Session 10

Modal Analysis of MDOF system Subjected to Dynamic force

By

Prof. M. G. Gadgil

Prof. Manohar G Gadgil



Prof. Manohar Gadgil is retired professor from VJTI. He was HOD of the structural department of VJTI.
He completed his Bachelor of Engineering in Civil from the University of Bombay in 1970 and M. Tech. in Structure from I.I.T. Powai in 1975.
He has published several papers at Indian and international conferences.
During the last 33 years, he has guided more than 100 P.G. students in their dissertation work.
The software needed for the projects was developed by him in the days when ready-to-use software was not available on the market.
He consults on several industry-sponsored projects like high-rise buildings, machine foundation equipment, industrial building structures, and many more.

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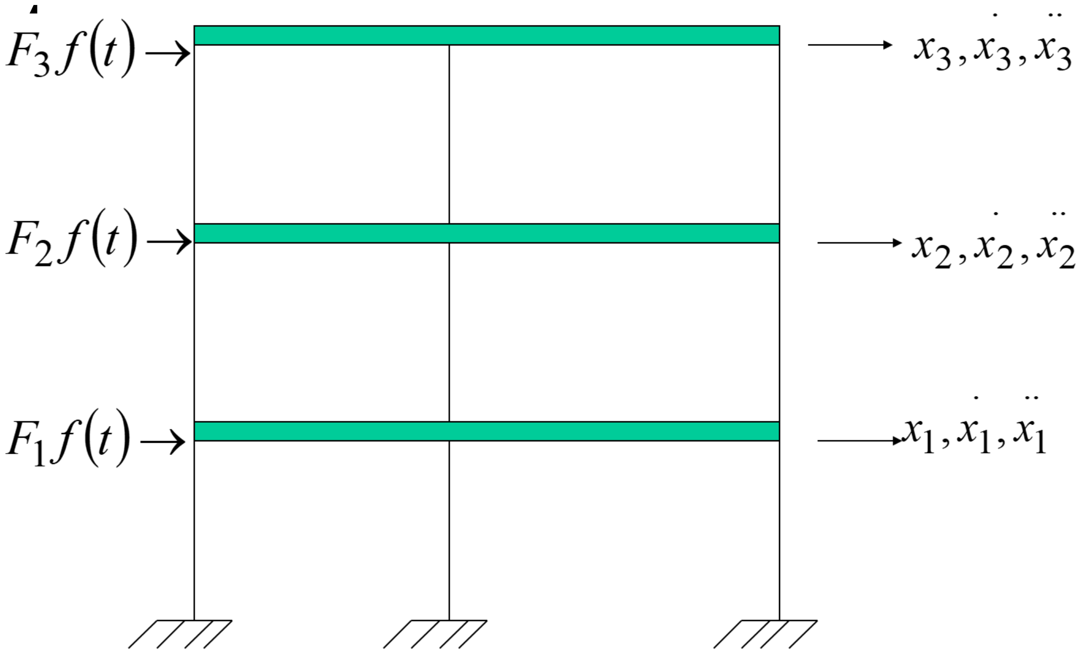

A typical MDOF system

f

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a

$$K_{11}x_1 + K_{12}x_2 + K_{13}x_3 + ... + K_{1n}x_n + M_1 \ddot{x}_1 = F_1 f(t)$$

$$K_{21}x_1 + K_{22}x_2 + K_{23}x_3 + ... + K_{2n}x_n + M_2 \ddot{x}_2 = F_2 f(t)$$

$$K_{31}x_1 + K_{32}x_2 + K_{33}x_3 + ... + K_{3n}x_n + M_3 \ddot{x}_1 = F_3 f(t)$$

Equation of motion of the system with no damping and no external force is given as

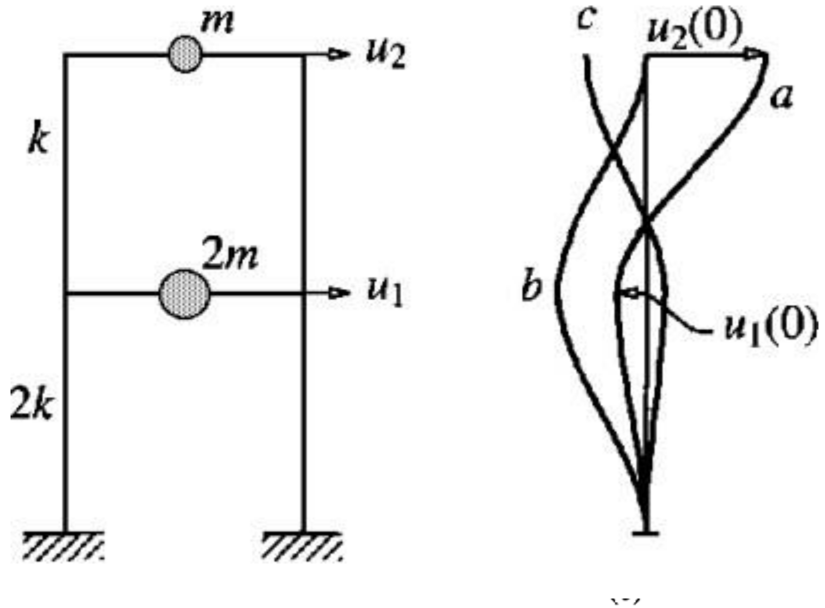
$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = 0$$

This represents a set of N homogenous ordinary differential equations of motion coupled through m and k matrices

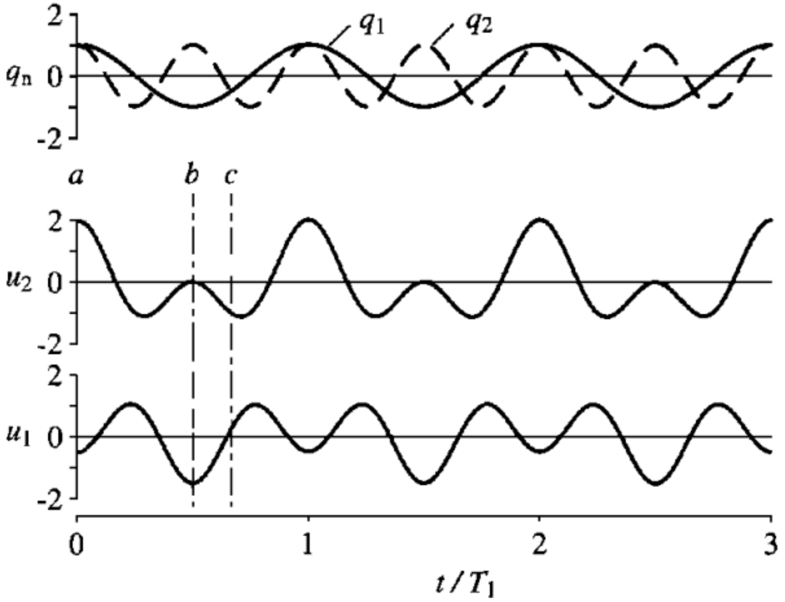
It required to solve the equations under initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0) \qquad \text{at } t = 0.$$

Instances a, b, c



Free vibration is initiated
With arbitrary initial
Displacements Shown
in curve a



Observed motion of mass m

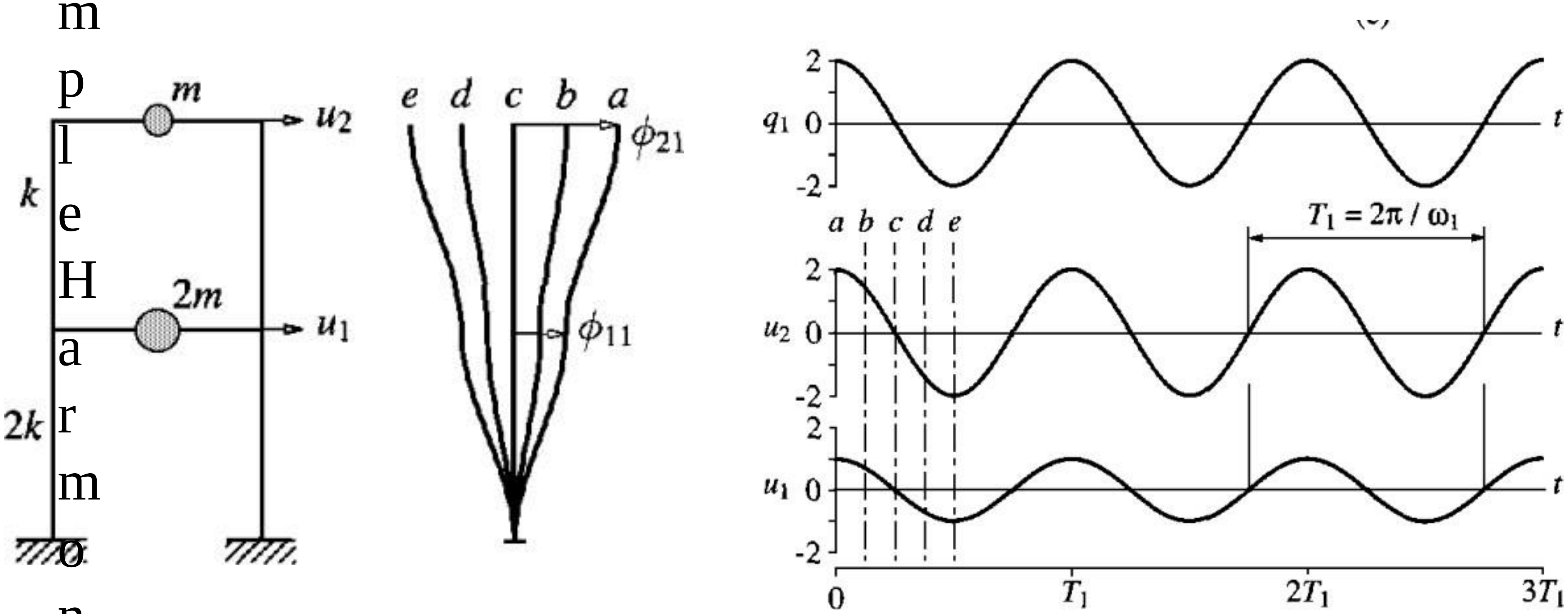
Observed motion of mass 2m

Under arbitrary displacements of each mass (initial condition)
We observe that the motion is not simple harmonic and

Frequency of mass can not be defined
Further, deflected shape (Ratio u1/u2) varies with time

On the other hand for SDOF system the motion is always harmonic when displaced by any arbitrary displacement

Simple Harmonic moti



Under these conditions of displacements

i. Both masses reach max disp at same time
ii. Both masses pass through equilibrium position at same time

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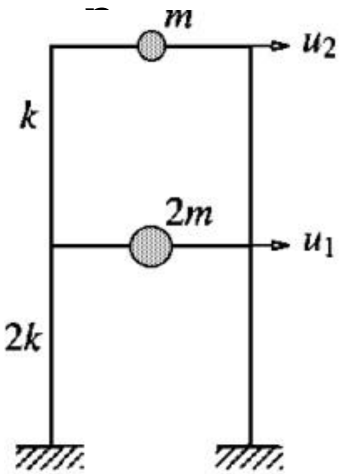

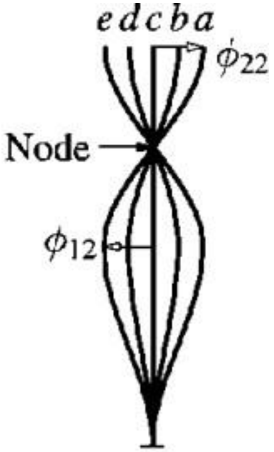

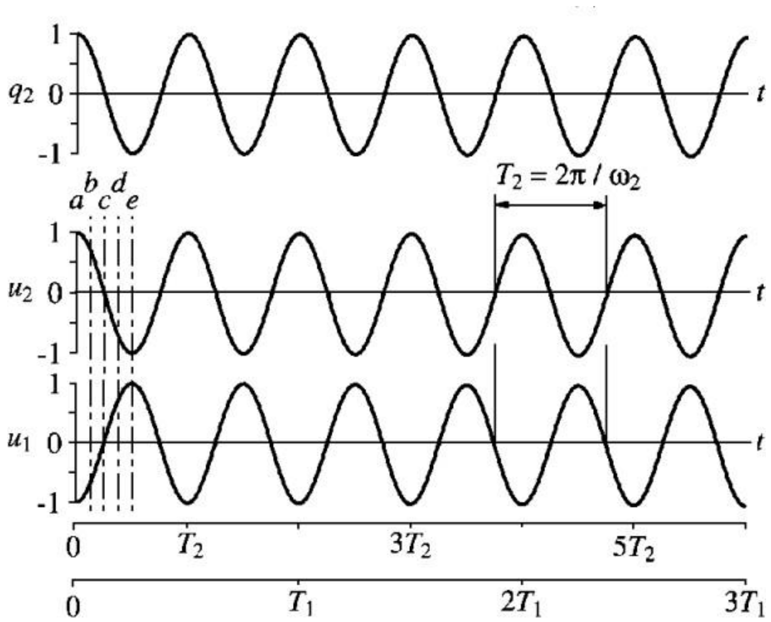

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A *natural period of vibration* T_n of an MDF system is the time required for one cycle of the simple harmonic motion in one of these natural modes. The corresponding *natural circular frequency of vibration* is ω_n and the *natural cyclic frequency of vibration* is f_n, where

m
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$$T_n = \frac{2\pi}{\omega_n} \qquad f_n = \frac{1}{T_n}$$

General comments on MDOF

- From the simple e

Natural vib

Free vibration of an undamped system in one of its natural Mode of vibration is given by

$$\mathbf{u}(t) = q_n(t)\phi_n$$

Time variation of displacement under simple harmonic motion is given as

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t$$

Natural vibration frequencies and modes contd

combined equation is written as

$$\mathbf{u}(t) = \phi_n \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right)$$

substituting this function in equation of motion for the entire structure we get

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = 0$$

$$\left[-\omega_n^2 \mathbf{m}\phi_n + \mathbf{k}\phi_n\right] q_n(t) = \mathbf{0}$$

Natural vib

Thus either

Either $q_n(t) = 0$, which implies that $\mathbf{u}(t) = \mathbf{0}$ and there is no motion of the system

or the natural modes ϕ_n and frequencies ω_n must satisfy the algebraic equation

$$\mathbf{k}\phi_n = \omega_n^2 \mathbf{m}\phi_n$$

Natural vibration frequencies and modes contd

Above equation is rewritten as

$$\left[\mathbf{k} - \omega_n^2 \mathbf{m}\right] \phi_n = \mathbf{0}$$

This is a set of N homogeneous algebraic equations. Non trivial solution is possible for these equation if if

$$\det\left[\mathbf{k} - \omega_n^2 \mathbf{m}\right] = 0$$

This is the so called characteristic / frequency equation

The determinant, on expansion , gives a polynomial of order N Giving N real and positive roots for natural frequency ω_n^2.

Positive roots of above equation are the N natural frequencies of vibration ω_n. for n = 1,2,3 …. N

They are also called Eigen values or characteristic values

For each value of ω_n we can solve homogenous equation

$$[\mathbf{k} - \omega_n^2 \mathbf{m}]\,\phi_n = \mathbf{0}$$

And obtain vector ϕ_n -- eigen vector which gives shape of the vibrating structure when one of the value is fixed as 1

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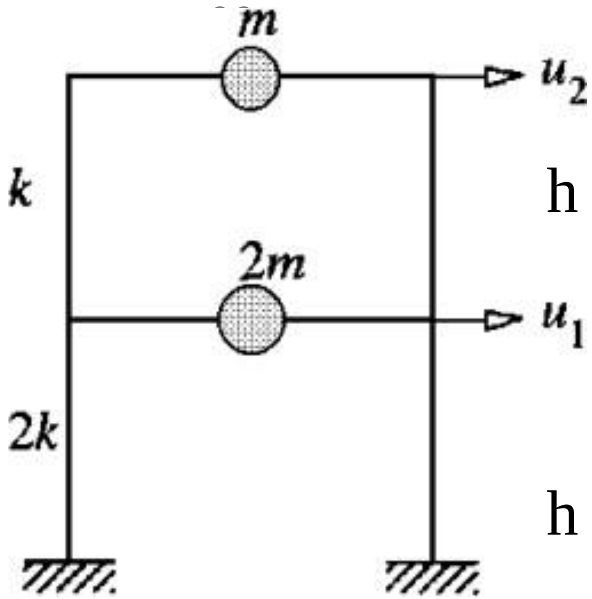

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Shear building

I of col = I_c

Mass matrix $\mathbf{m} = \begin{bmatrix} 2m & \\ & m \end{bmatrix}$

Stiffness matrix $\mathbf{k} = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}$

where $k = 24EI_c/h^3$

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = 0$$

Subbstituting K and M in following equation

$$\det\left[\mathbf{k} - \omega_n^2 \mathbf{m}\right] = 0$$

We get following polynomial equation

$$(2m^2)\omega^4 + (-5km)\omega^2 + 2k^2 = 0$$

The two roots are $\omega_1^2 = k/2m$ and $\omega_2^2 = 2k/m$, and the two natural frequencies are

$$\omega_1 = \sqrt{\frac{k}{2m}} \qquad \omega_2 = \sqrt{\frac{2k}{m}}$$

Substituting for k gives

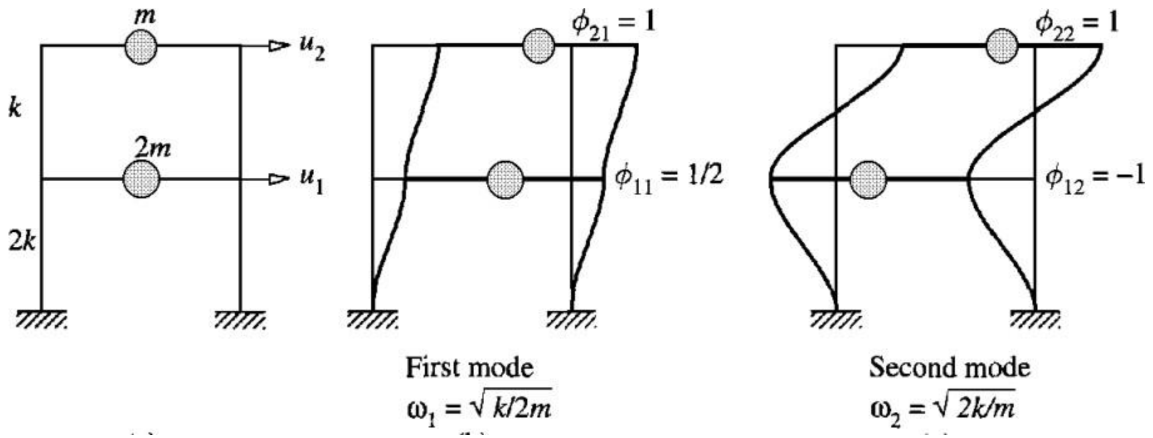
$$\omega_1 = 3.464\sqrt{\frac{EI_c}{mh^3}} \qquad \omega_2 = 6.928\sqrt{\frac{EI_c}{mh^3}}$$

For each of these values we solve the following equation

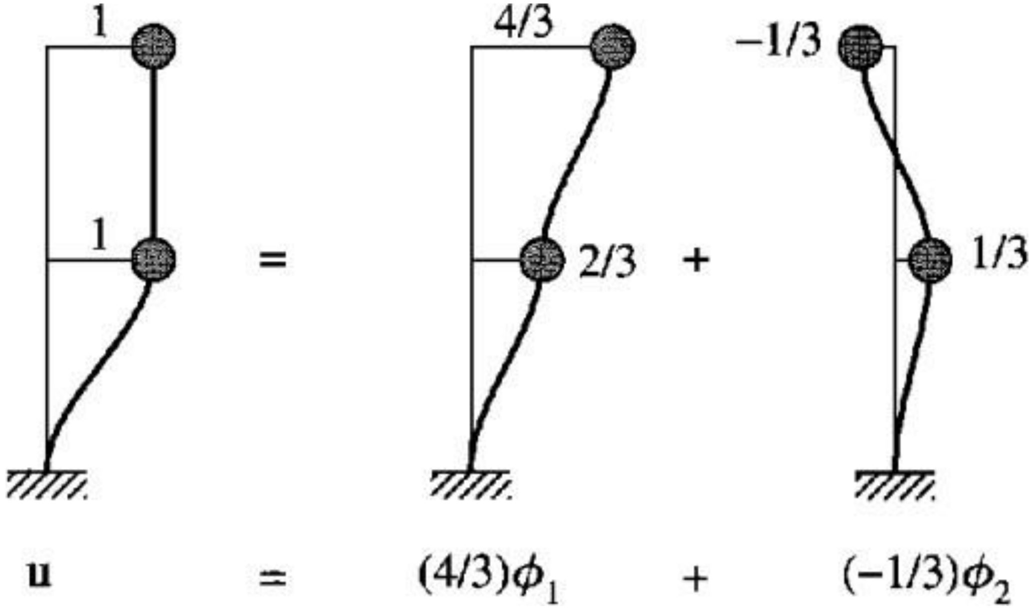
$$\left[\mathbf{k} - \omega_n^2\mathbf{m}\right]\phi_n = \mathbf{0}$$

We get eigen vectors as

$$\phi_1 = \begin{Bmatrix} \frac{1}{2} \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$



Modal expansion of displacements

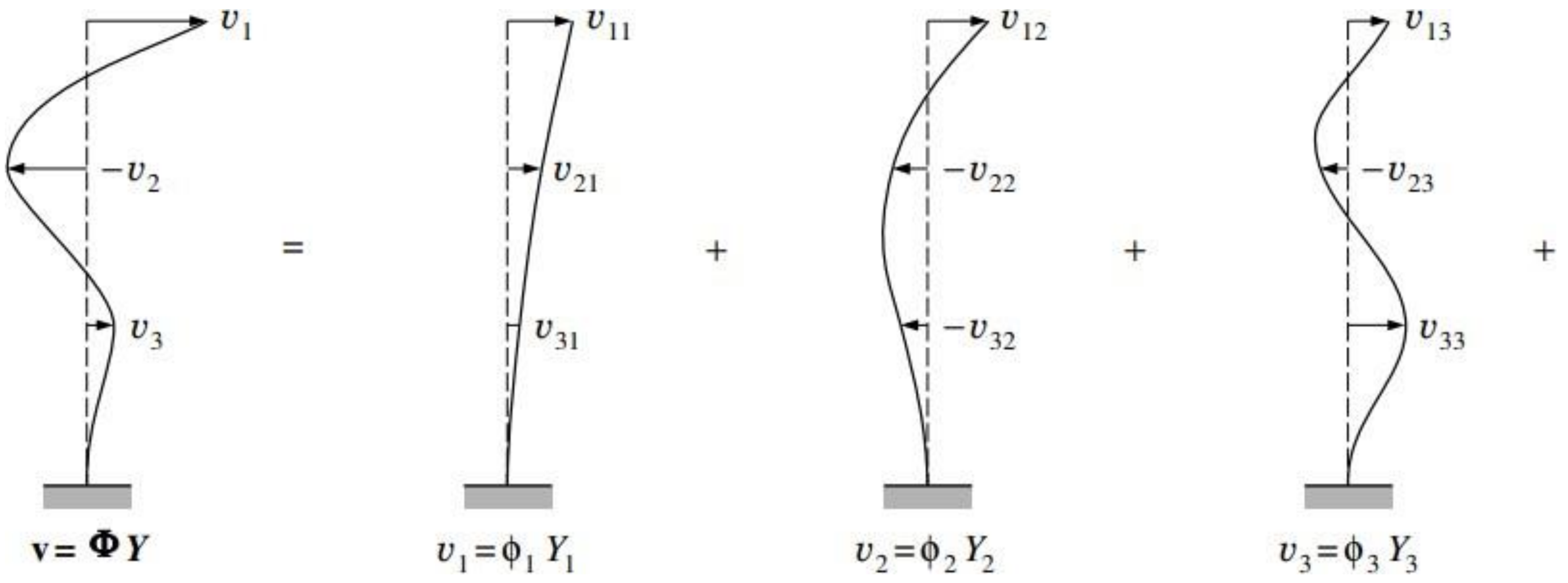


$U_1 = q_1\varphi_{11} + q_2\varphi_{12}$
$U_2 = q_1\varphi_{21} + q_2\varphi_{22}$

$\Phi_{11}=1/2,\quad \Phi_{21}=1,\quad q_1=4/3,$

$\Phi_{12}=-1,\quad \Phi_{22}=1,\quad q_2=-1/3,$

Modal expansion of displacements contd.



$$v_1 = Y_1 \emptyset_{11} + Y_2 \emptyset_{12} + Y_3 \emptyset_{13}$$

$$v_2 = Y_1 \emptyset_{21} + Y_2 \emptyset_{22} + Y_3 \emptyset_{23}$$

$$v_3 = Y_1 \emptyset_{31} + Y_2 \emptyset_{32} - Y_3 \emptyset_{33}$$

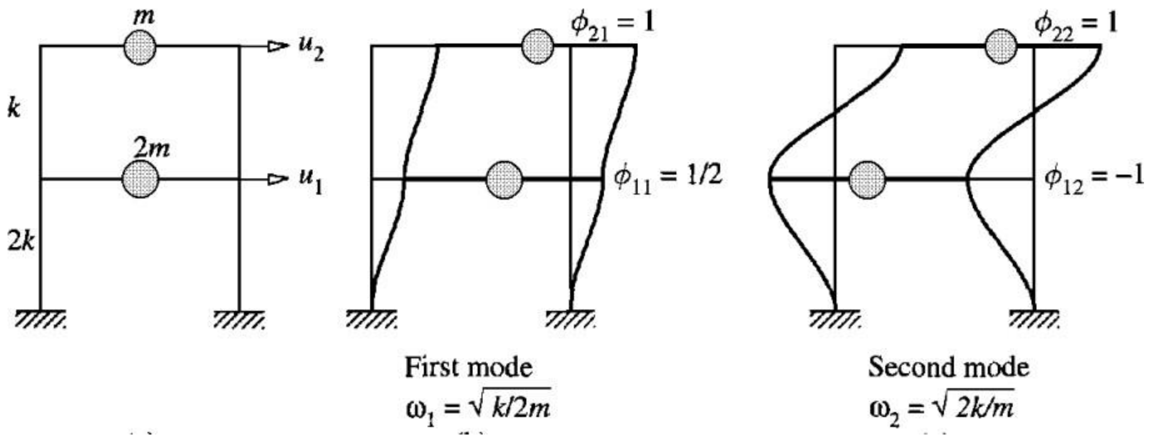
Orthogonal

Natural modes corresponding to different natural frequencies can be shown to satisfy following orthogonality condition wrt mass and stiffness

$$\phi_n^T \mathbf{k} \phi_r = 0 \qquad \phi_n^T \mathbf{m} \phi_r = 0$$

We shall verify the same with a 2 DOF system and then prove it for general case

Verification of orthogonality of modes 1 and 2



$$\emptyset_{11} M_1 \emptyset_{21} = \frac{1}{2} x 2m x (-1) = -m$$

$$\emptyset_{21} M_2 \emptyset_{22} = 1 \; x \; m \; x \; (1) = m$$

summation is zero

Proof of orthogonality of any two distinct modes of natural vibbration

Betty maxwel’s reciprocal theorem

Work done by one set of forces on the deflections caused by Other set of forces

is equal to

Work done by other set of Forces on the deflection caused by first set of forces

For system under natural vibration only inertia forces are Acting on the system

System vibrating in mode no m, having r no of masses have following parameters

Masses $m_1, m_2, m_3, \ldots . m_r$

Deflections $\emptyset_{1m}, \emptyset_{2m}, \emptyset_{3m} \ldots . \emptyset_{rm}$

Inertia forces $m_1 \emptyset_{1m} \omega_m^2, \; m_2 \emptyset_{2m} \omega_m^2, m_3 \emptyset_{3m} \omega_m^2 \ldots m_r \emptyset_{rm} \omega_m^2$

System vibrating in mode no n, having r no of masses have following parameters

Masses $m_1, m_2, m_3, \ldots . m_r$

Deflections $\emptyset_{1n}, \emptyset_{2n}, \emptyset_{3n} \ldots . \emptyset_{rn}$

Inertia forces $m_1 \emptyset_{1n} \omega_n^2,\ m_2 \emptyset_{2n} \omega_n^2, m_3 \emptyset_{3n} \omega_n^2 \ldots m_r \emptyset_{rn} \omega_n^2$

Using Betty Maxwell's reciprocal theorem

Work done by r forces of mode m on deflections of mode n

$$m_1 \emptyset_{1m} \omega_m^2 \ \text{X} \ \emptyset_{1n} + m_2 \emptyset_{2m} \omega_m^2 \ \text{X} \ \emptyset_{2n} + \text{----} + m_r \emptyset_{rm} \omega_m^2 \ \text{X} \ \emptyset_{rn}$$

$$= \omega_m^2 \sum_{r=1}^{r=no\ of\ masses} m_r \emptyset_{rm} \emptyset_{rn}$$

=Work done by r forces of mode n on deflections of mode m

$$= m_1 \ \emptyset_{1n} \omega_n^2 \ \text{X} \ \emptyset_{1m} + m_2 \emptyset_{2n} \omega_n^2 \ \text{X} \ \emptyset_{2m} + \text{----} + m_r \emptyset_{rn} \omega_n^2 \ \text{X} \ \emptyset_{rm}$$

$$= \omega_n^2 \sum_{r=1}^{r=no\ of\ masses} m_r \emptyset_{rn} \emptyset_{rm}$$

By Betty maxwel's reciprocal theorem

$$\omega_m^2 \sum_{r=1}^{r=no\ of\ masses} m_r \emptyset_{rm} \emptyset_{rn} = \omega_n^2 \sum_{r=1}^{r=no\ of\ masses} m_r \emptyset_{rn} \emptyset_{rm}$$

$$(\omega_m^2 - \omega_n^2) \sum_{r=1}^{r=no\ of\ masses} m_r \emptyset_{rm} \emptyset_{rn} = 0$$

Giving the orthogonality principle when $(\omega_m^2 \neq \omega_n^2)$

$$\sum_{r=1}^{r=no\ of\ masses} m_r \emptyset_{rm} \emptyset_{rn} = 0$$

Modal vectors

A modal displacement shape is considered as a vector with r no of components (one for each DOF)
Thus for modes m and n we have modal (shape) vectors as

$$\begin{Bmatrix} \phi_{1m} \\ \phi_{2m} \\ \phi_{3m} \\ . \\ . \\ . \\ \phi_{rm} \end{Bmatrix} \qquad \begin{Bmatrix} \phi_{1n} \\ \phi_{2n} \\ \phi_{3n} \\ . \\ . \\ . \\ \phi_{rn} \end{Bmatrix}$$

For geometric orthogonality in 3D space for two vectors (say forces) F_{1x} F_{1y} F_{1z} and F_{2x} F_{2y} F_{2z} we prove orthogonality by taking the dot product of the two vectors

$$F_{1x} F_{2x} + F_{1y} F_{2y} + F_{1z} F_{2z} = 0$$

Concept of modal analysis of MDOF system

General equation of motion of MDOF system (without damping) subjected to any dynamic force is given by

$$[M]\ddot{U}+[K]\boldsymbol{U}=\{\boldsymbol{p}(\boldsymbol{t})\}$$

[M] is n x n size mass matrix
[K] is n x n size stiffness matrix
{p(t)} is n x 1 size force matrix
It is extremely difficult to solve these coupled simultaneous Differential equations when DOF is a large no.
Better option is to convert these equations into modal equations And then solve such simple equations for each mode of natural vibration

Concept of m

$$[M]\ddot{U}+[K]U=\{\boldsymbol{p}(\boldsymbol{t})\}$$

In this equation we substitute the modal contribution as

$$\mathbf{u}(t)=\sum_{r=1}^{N}\phi_r q_r(t)=\mathbf{\Phi q}(t)$$

φ_r= mode shape for r th mode

$q_{r(}(t)$=weightage given for r th mode at time t

Concept of modal analysis of MDOF system contd

$[M]\ddot{U}+[K]U=\{\boldsymbol{p}(\boldsymbol{t})\}$ use $\mathbf{u}(t) = \sum_{r=1}^{N} \phi_r q_r(t) = \mathbf{\Phi q}(t)$

Substituting value of U and $\ddot{U}$ we now get

$$\sum_{r=1}^{N} \mathbf{m}\,\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \mathbf{k}\,\phi_r q_r(t) = \mathbf{p}(t)$$

Premultiplying each term in this equation by ϕ_n^T gives

$$\sum_{r=1}^{N} \phi_n^T\,\mathbf{m}\,\phi_r\,\ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T\,\mathbf{k}\,\phi_r\,q_r(t) = \phi_n^T\,\mathbf{p}(t)$$

Concept of m

Because of the orthogonality relations all terms in each of the summations vanish, except the $r = n$ term, reducing this equation to

$$(\phi_n^T \mathbf{m} \phi_n) \ddot{q}_n(t) + (\phi_n^T \mathbf{k} \phi_n) q_n(t) = \phi_n^T \mathbf{p}(t)$$

or

$$M_n \ddot{q}_n(t) + K_n q_n(t) = P_n(t)$$

where

$M_n = \phi_n^T \mathbf{m} \phi_n$ Is the modal mass Or generalized mass

$K_n = \phi_n^T \mathbf{k} \phi_n$ Is the modal stiffness Or generalized stiffness

$P_n(t) = \phi_n^T \mathbf{p}(t)$ Is the modal force Or generalized force

For mode n

$$M_n \ddot{q}_n(t) + K_n q_n(t) = P_n(t)$$ Eq. of motion for n th mode

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$ Eq. of motion for SDOF

The two equations are similar
the modal equation for n th mode will give solution for q_n for n th mode.
Displacements of other mass points can be determined from mode shape

Such transformation can be done for each mode n = 1,2,3 --- n

Thus a set of N coupled differential equation is transformed into N uncoupled differential equations in modal coordinates q_n n = 1,2,3--n

Modal Equation of motion for ground motion as input is given by

Equation of motion for MDOF subjected to ground motion is given by

$$M\left(\ddot{X}+\ddot{X}_g\right)+\boldsymbol{KX}=\mathbf{0}$$

$$M\ddot{X}+\boldsymbol{KX}=-M\ddot{X}_g$$

Using modal split up of general displacement X

$$\boldsymbol{M}\ddot{\boldsymbol{Y}}\varphi+\boldsymbol{KY}\varphi=-M\ddot{X}_g$$

where $\boldsymbol{Y}\varphi=\boldsymbol{Y}_{\mathbf{1}}\varphi_1+\boldsymbol{Y}_{\mathbf{2}}\varphi_2+\boldsymbol{Y}_{\mathbf{3}}\varphi_3+\ldots.$

and $\ddot{\boldsymbol{Y}}\varphi=\ddot{\boldsymbol{Y}}_{\mathbf{1}}\varphi_1+\ddot{\boldsymbol{Y}}_{\mathbf{2}}\varphi_2+\ddot{\boldsymbol{Y}}_{\mathbf{3}}\varphi_3+\ldots.$

Premultiplying both sides by φ_n^T

• Using orth

$$\varphi_n^T \boldsymbol{M} \ddot{\boldsymbol{Y}} \varphi + \varphi_n^T \boldsymbol{K} \boldsymbol{Y} \varphi = -\varphi_n^T \mathrm{M} \ddot{\mathrm{X}}_{\boldsymbol{g}}$$

$$\varphi_n^T \boldsymbol{M} \ddot{\boldsymbol{Y}}_{\boldsymbol{n}} \varphi_{\boldsymbol{n}} + \varphi_n^T \boldsymbol{K} \boldsymbol{Y}_{\boldsymbol{n}} \varphi_{\boldsymbol{n}} = -\varphi_n^T \mathrm{M} \ddot{\mathrm{X}}_{\boldsymbol{g}}$$

which we can write as

$$\mathrm{M}_{n,eq} \ddot{\boldsymbol{Y}}_{\boldsymbol{n}} + \mathrm{K}_{n,eq} \boldsymbol{Y}_{\boldsymbol{n}} = -\varphi_n^T \mathrm{M} \ddot{\mathrm{X}}_{\boldsymbol{g}} = -\varphi_n^T \mathrm{M} \ddot{\mathrm{U}}_{\boldsymbol{go}} \, \boldsymbol{f}(\boldsymbol{t})$$

where

$$M^{n}_{eq} = \varphi^{T}_{n} M \boldsymbol{\varphi_n} = \text{Equivalent Mass}$$

$$K^{n}_{eq} = \varphi^{T}_{n} K \boldsymbol{\varphi_n} = \text{Equivalent Stiffness}$$

and

$$\varphi^{T}_{n} M = \sum_{r=1}^{\text{no of masses}} M_{r} \boldsymbol{\varphi_{rn}}$$

Modal Response is Given by

D_n = Modal Static deflection X DLF

$$= \frac{\sum M_r \phi_{rn} \ddot{u}_g}{\sum M_r \phi_{rn}^2 \omega_n^2} x DLF_n$$

$$= \frac{\sum M_r \phi_{rn}}{\sum m_r \phi_{rn}^2 \omega_n^2} \ddot{u}_g \, x DLF_n$$

Displacement Response at any mass point r in original structure is given by

$$D_{rn} = \frac{\sum M_r \phi_{rn}}{\sum M_r \phi^2{rn}} \frac{\ddot{u}g}{\omega_n^2} x \phi_{rn} DLF_n$$

Pseudo acceleration Response at any mass point r in original structure is given by

$$D_r^{**n} = D_r^n \times \omega_n^2 = \frac{\sum M_r \phi_{rn}}{\sum M_r \phi^2 rn} \ddot{u}g \, x \phi_{rn} DLF_n$$

Pseudo Inertia force at any mass point r in original structure is given by

$$F_r^n = D_r^{**n} \times M_r$$

$$= M_r \frac{\sum M_r \phi_{rn}}{\sum M_r \phi^2_{rn}} \ddot{u}g \, x \phi_{rn} DLF_n$$

Seismic force as per IS 1893-2016

$$Q_{ik} = A_k \phi_{ik} P_k W_i$$

$$A_h = \frac{\left(\frac{Z}{2}\right)\left(\frac{S_a}{g}\right)}{\left(\frac{R}{I}\right)}$$

$$P_k = \frac{\sum_{i=1}^{n} W_i \phi_{ik}}{\sum_{i=1}^{n} W_i (\phi_{ik})^2}$$

$$Q_{ik} = \frac{\sum_{i=1}^{n} W_i \phi_{ik}}{\sum_{i=1}^{n} W_i (\phi_{ik})^2} \; \phi_{ik} \; \frac{\left(\frac{Z}{2}\right)\left(\frac{S_a}{g}\right)}{\left(\frac{R}{I}\right)} \; W_i$$

As per IS code

$$\frac{\sum M_r \phi_{rn}}{\sum M_r \phi^2 rn} \; \phi_{rn} \; (\ddot{u}g \;\; DLF_n) \; \mathbf{M_r}$$

As per theory

Calculations o

- SDOF

$$Y_{dynamic} = Y_{Static} * DLF$$

$$Y_{static} = \frac{F_o}{K}$$

DLF – – – function of

T and f(t)

- MDOF

$$Y_{dynamic}^{(n)} = Y_{Static}^{(n)} * DLF^{(n)}$$

$$Y_{static}^{(n)} = \frac{F_{eq}^{(n)}}{K_{eq}^{n}} =$$

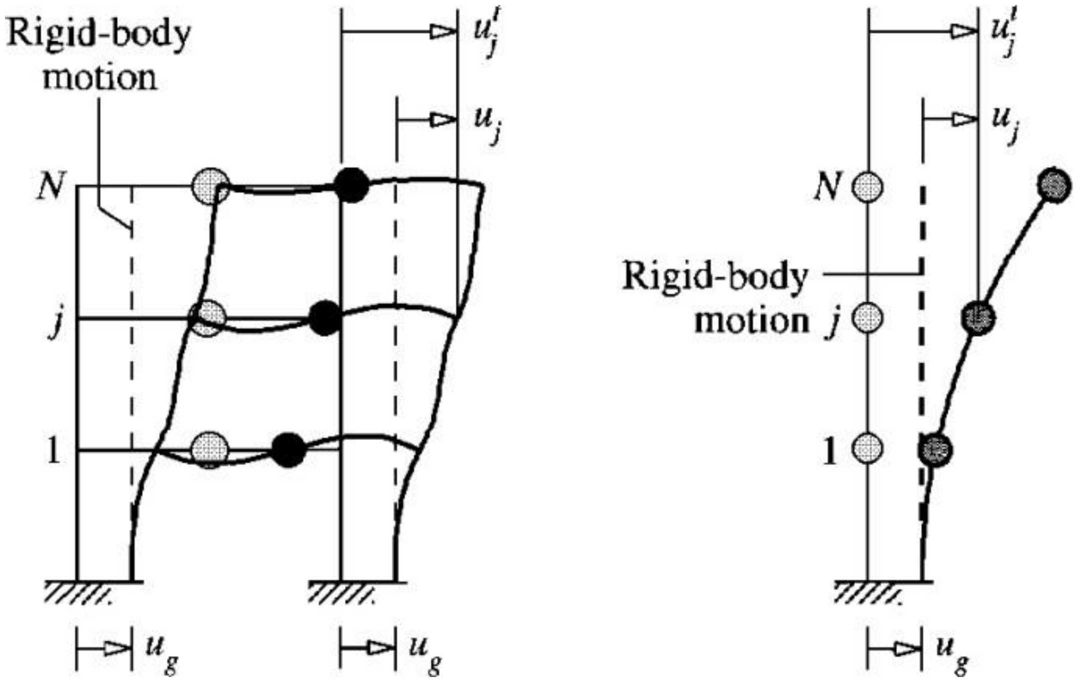
$$= \frac{F_{eq}^{(n)}}{M_{eq}^{n} \omega^2}$$

DLF^n – – – function of

T_n and f(t)

Seismic analysis of MDOF system RHA

We now consider a tower and a typical building frame subjected to ground motion



At any instant displacement is given as $u_j^t(t) = u_j(t) + u_g(t)$

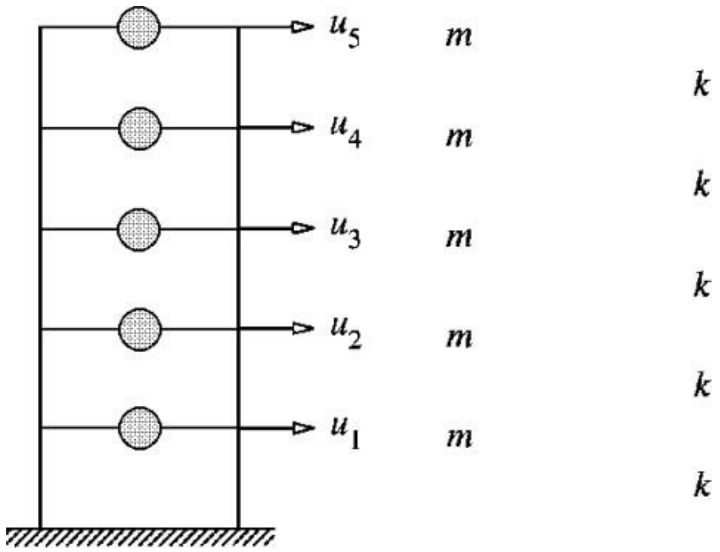
Seismic analysis of MDOF system RHA contd

Equation of equilibrium in dynamic state is given as

$$f_i + f_s = 0$$

$$M(\ddot{u} + \ddot{u}_g) - Ku = 0$$

$$M\ddot{u} + Ku = -M\ddot{u}_g = -M\ddot{u}_{g0}(t)$$



5 storey plane frame

Structure property matrix and free vibration characteristics

The mass and stiffness matrices of the structure are

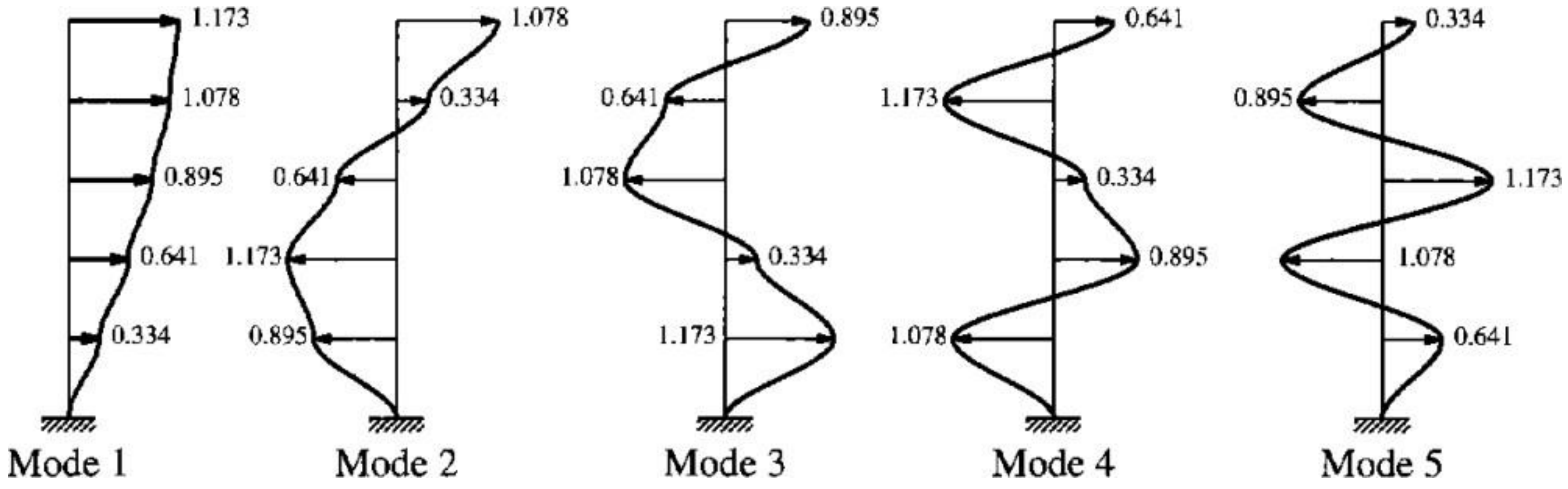
$$\mathbf{m} = m\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

Determined by solving the eigenvalue problem, the natural frequencies are

$$\omega_n = \alpha_n \left(\frac{k}{m}\right)^{1/2}$$

where $\alpha_1 = 0.285$, $\alpha_2 = 0.831$, $\alpha_3 = 1.310$, $\alpha_4 = 1.682$, and $\alpha_5 = 1.919$.

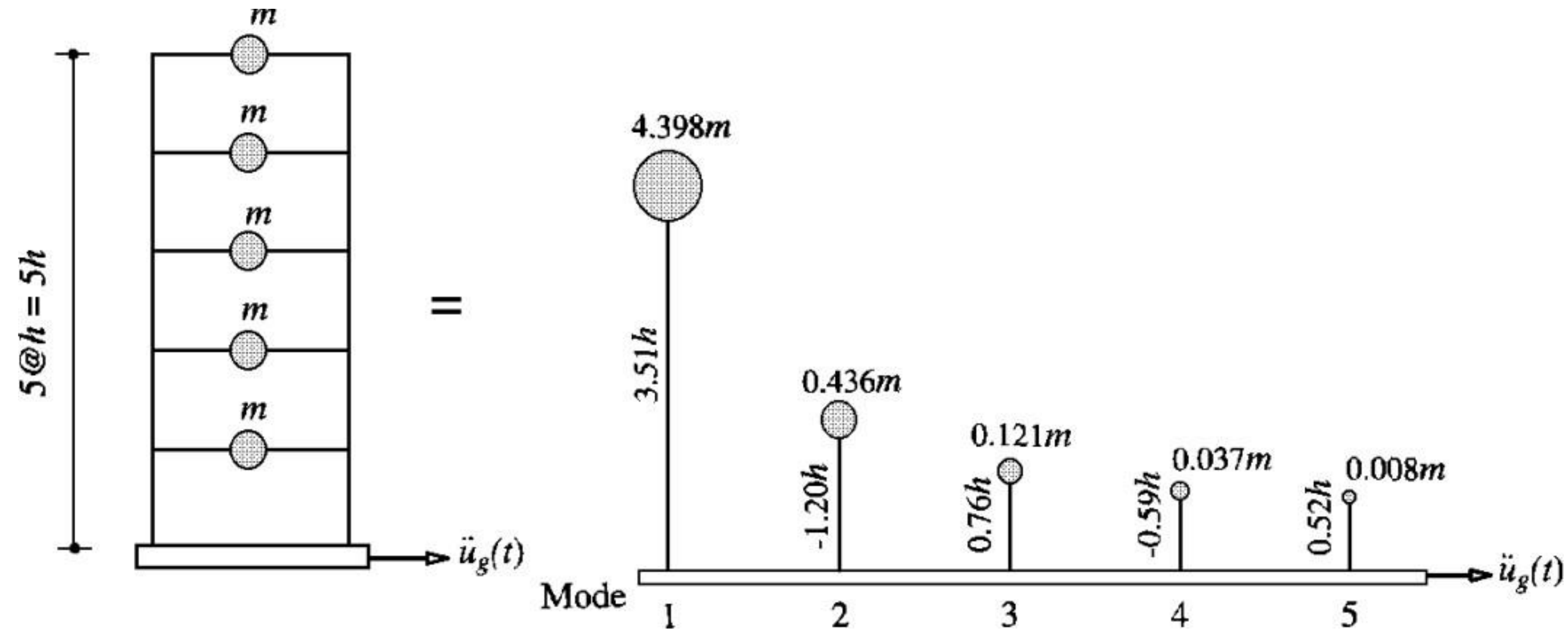
Structure property matrix and free vibration characteristics contd.



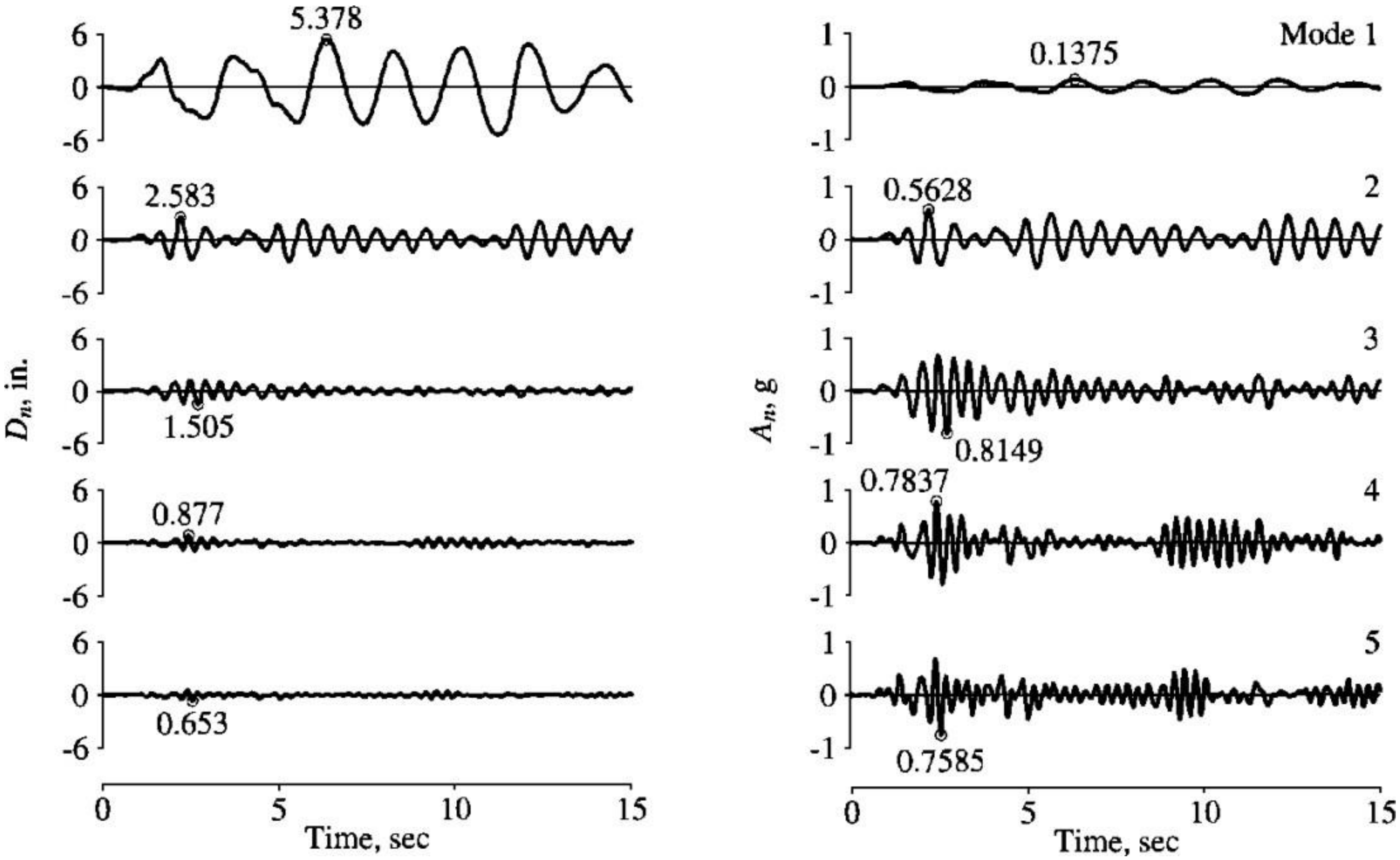
Natural modes of vibration of uniform five-story shear building.

Equivalent SDOF systems

Effective modal masses and effective modal heights.

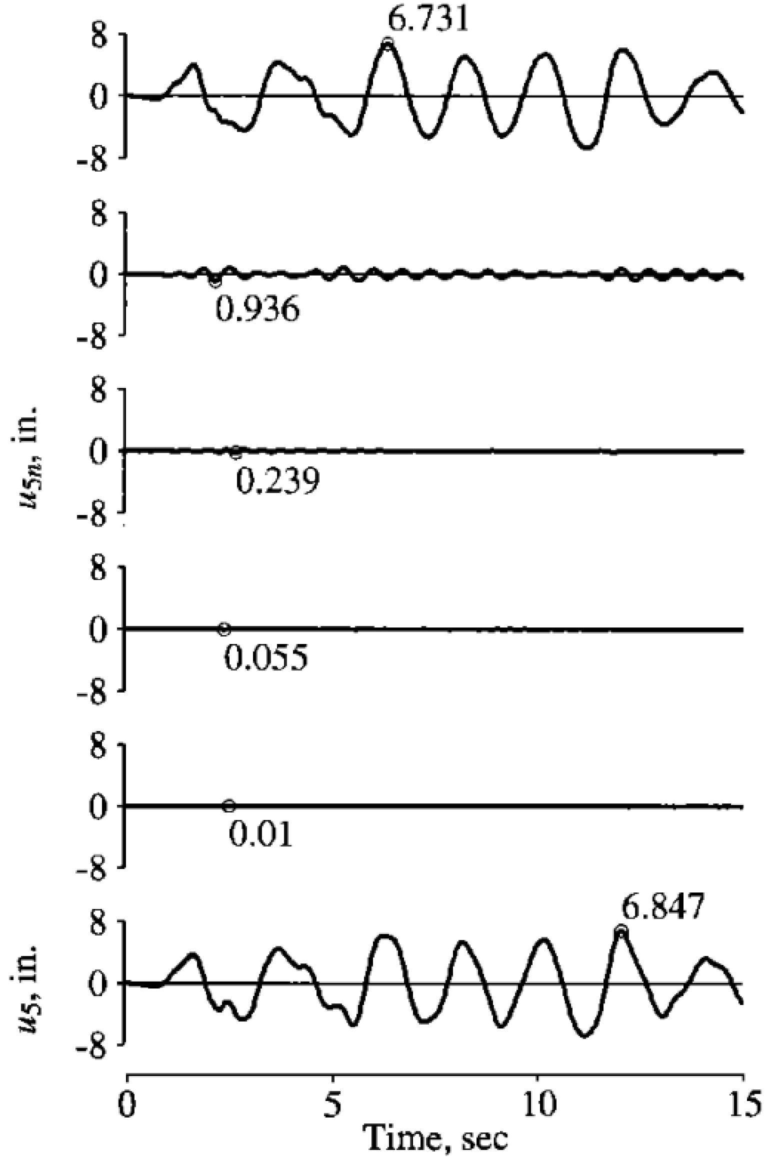


Response history analysis for El centro N-S ground motion



Displacement $D_n(t)$ and pseudo-acceleration $A_n(t)$ responses of modal SDF systems.

Total response history representation for top floor displacement



Roof displacement

modal contributions, $u_{5n}(t)$

and total responses, $u_5(t)$

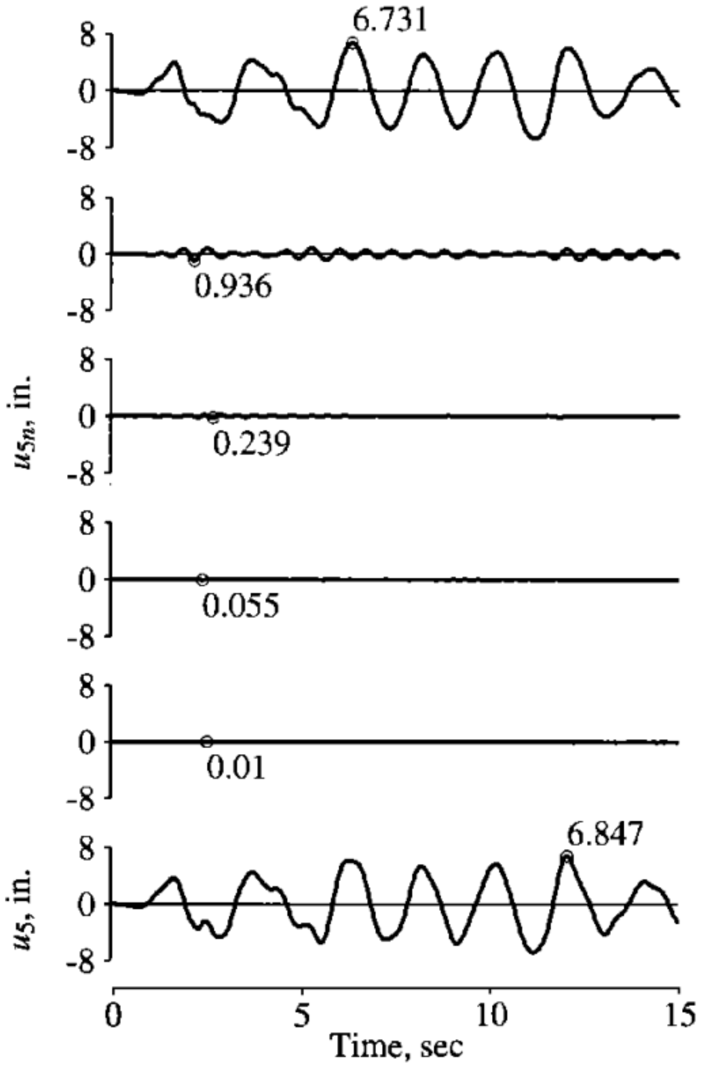
RESPONSE SPECTRUM ANALYSIS

Response history analysis presents structural response as a function of time but structural design is usually based peak values of forces and displacements over the duration of earthquake

Peak response of SDOF system can be accurately determined by using response spectrum for a given ground motion as the response spectrum is drawn using a SDOF system only

For MDOF system, there are some additional concepts which need to be used to get total (maximum) response of the structure

Timing of peak response in RHA



Modal and total response

Modal combination of response

In response spectrum, only peaks are collected from different modes

For different modes, peaks are reached at different times , during the earthquake

Even the combined response reaches maximum at yet another time

Approximations must be introduced in combining the peak modal responses r_{no} determined from the earthquake response spectrum because no information is available when these peak modal values occur.

Modal combination of response contd.

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Modal combination of response contd.

All peaks occurring at same time with same sign is an upper bound on the solution

Thus actual response is always less than this upperbound

$$r_o \leq \sum_{n=1}^{N} |r_{no}|$$

Modal combination of response contd.

- The square root

$$r_o \simeq \left(\sum_{n=1}^{N} r_{no}^2 \right)^{1/2}$$

Modal combination of response contd.

Complete quadratic combination (CQC) rule is a modal combination rule applicable to a more wider variety of problems

$$r_o \simeq \left(\sum_{i=1}^{N} \sum_{n=1}^{N} \rho_{in} r_{io} r_{no} \right)^{1/2}$$

r_o =Peak response at any mass point

r_{io} =Peak response at that mass point in mode i

r_{no} =Peak response at that mass point in mode n

= Corelation coeff between mode i and mode n

$$\rho_{in} = \frac{8\zeta^2(1+\beta_{in})\beta_{in}^{3/2}}{(1-\beta_{in}^2)^2 + 4\zeta^2\beta_{in}(1+\beta_{in})^2}$$

ζ = modal damping, generally 5 %

β_{in} = frequency ratio $= \omega_i/\omega_n$

IS 1893-2015 gives same equations

$$\lambda = \sqrt{\sum_{i=1}^{N_m}\sum_{j=1}^{N_m} \lambda_i \; \rho_{ij} \; \lambda_j}$$

where

λ = estimate of peak response quantity;

λ_i = response quantity in mode i (with sign);

λ_j = response quantity in mode j (with sign);

ρ_{ij} = cross-modal correlation co-efficient

$$= \frac{8\;\zeta^2(1+\beta)\;\beta^{1.5}}{\left(1-\beta^2\right)^2 + 4\;\;\zeta^2\beta\;\;(1+\beta)^2};$$

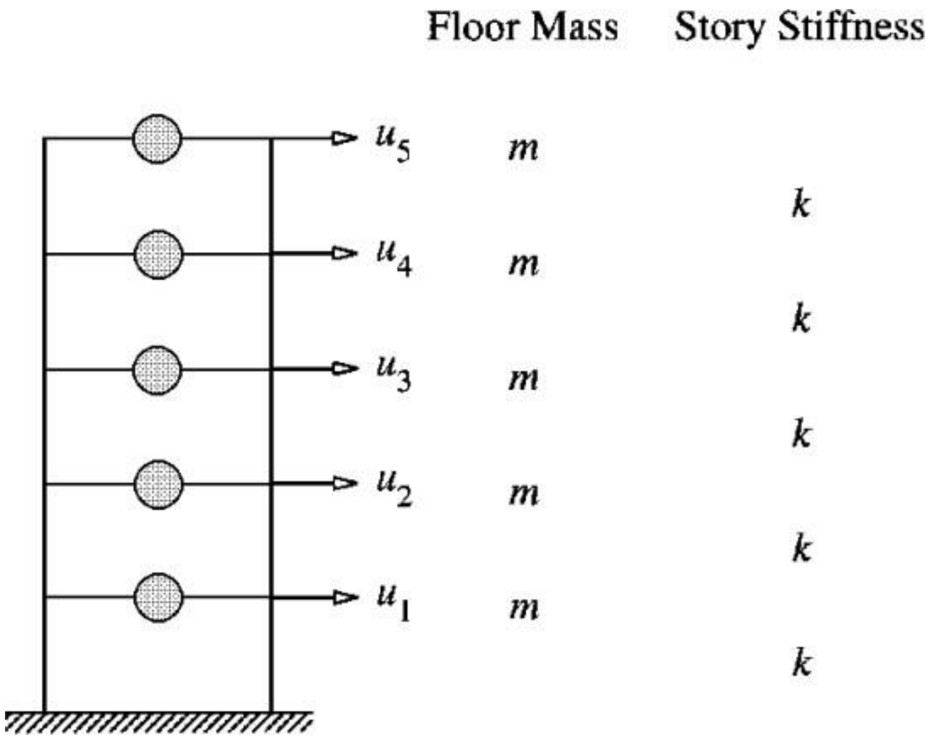
β = natural frequency ratio = $\dfrac{\omega_j}{\omega_i}$;

ζ = modal damping coefficient ratio which shall be taken as 0.05;

R

e

s

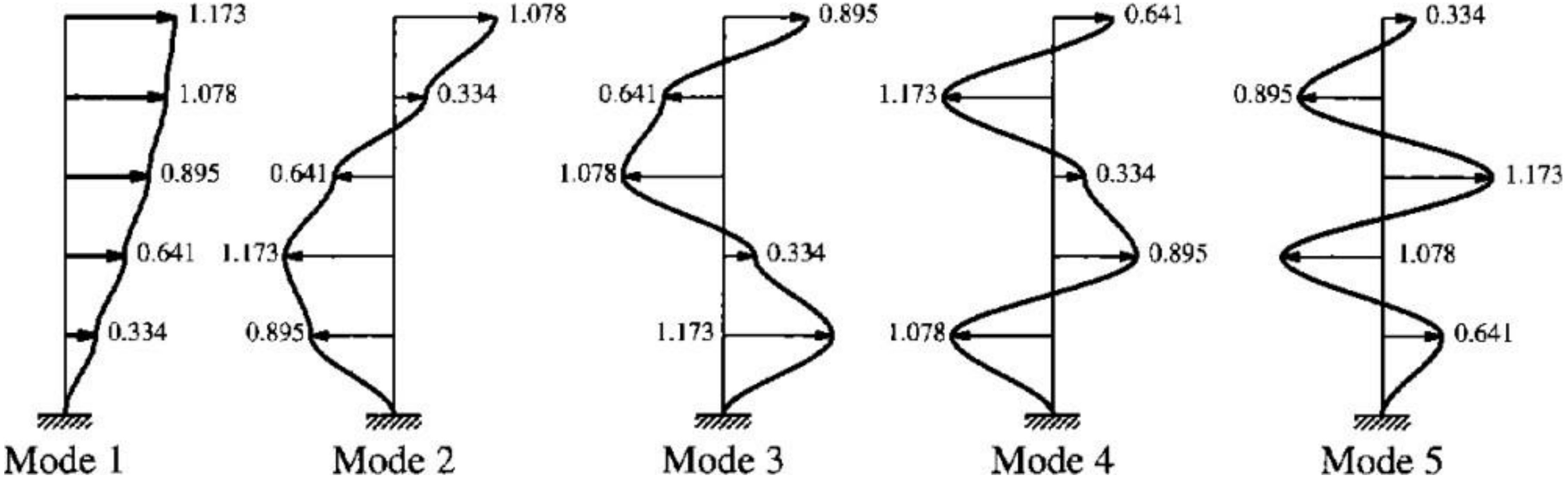


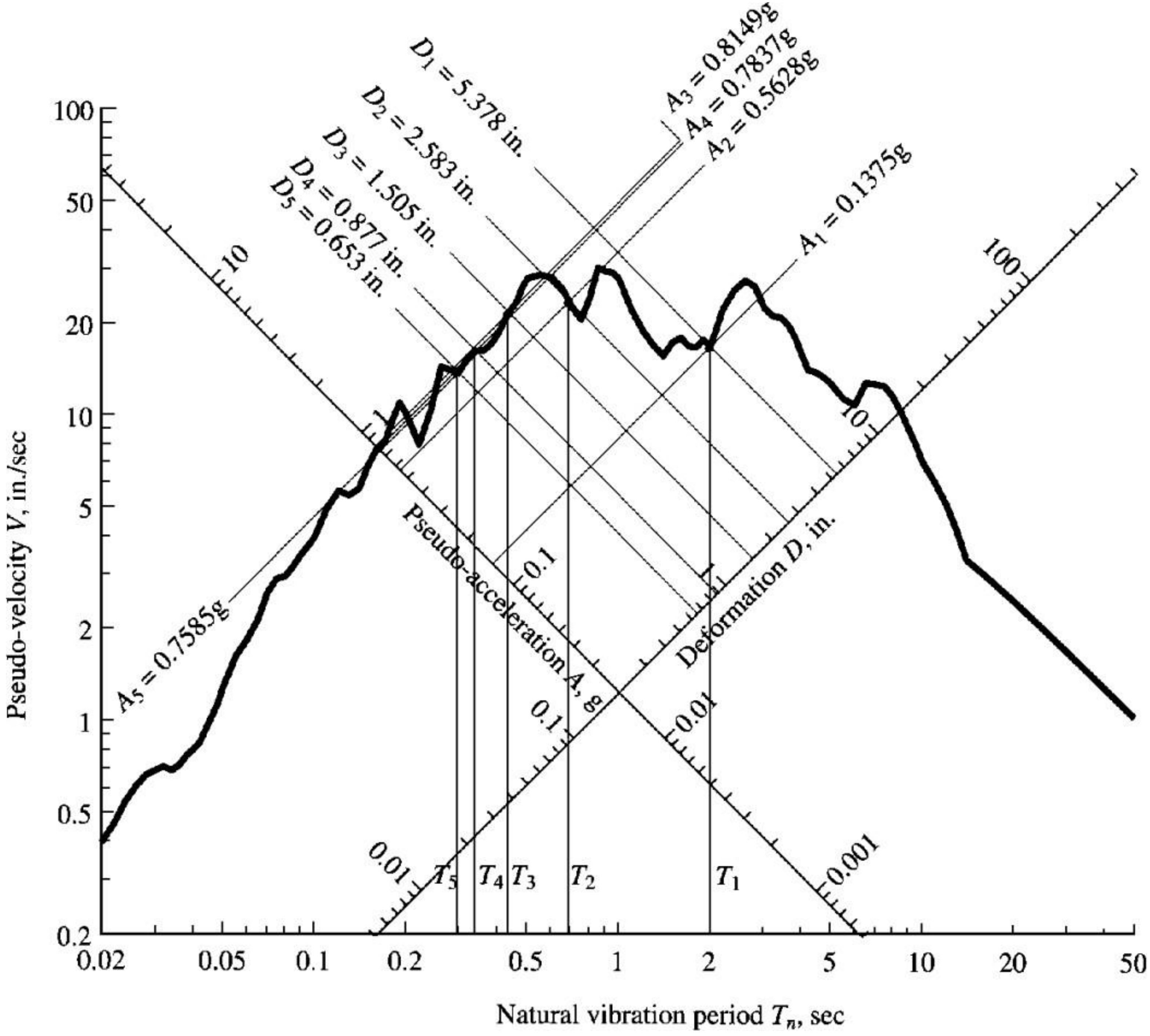
s

p

$$\omega_n = \alpha_n \left(\frac{k}{m}\right)^{1/2}$$

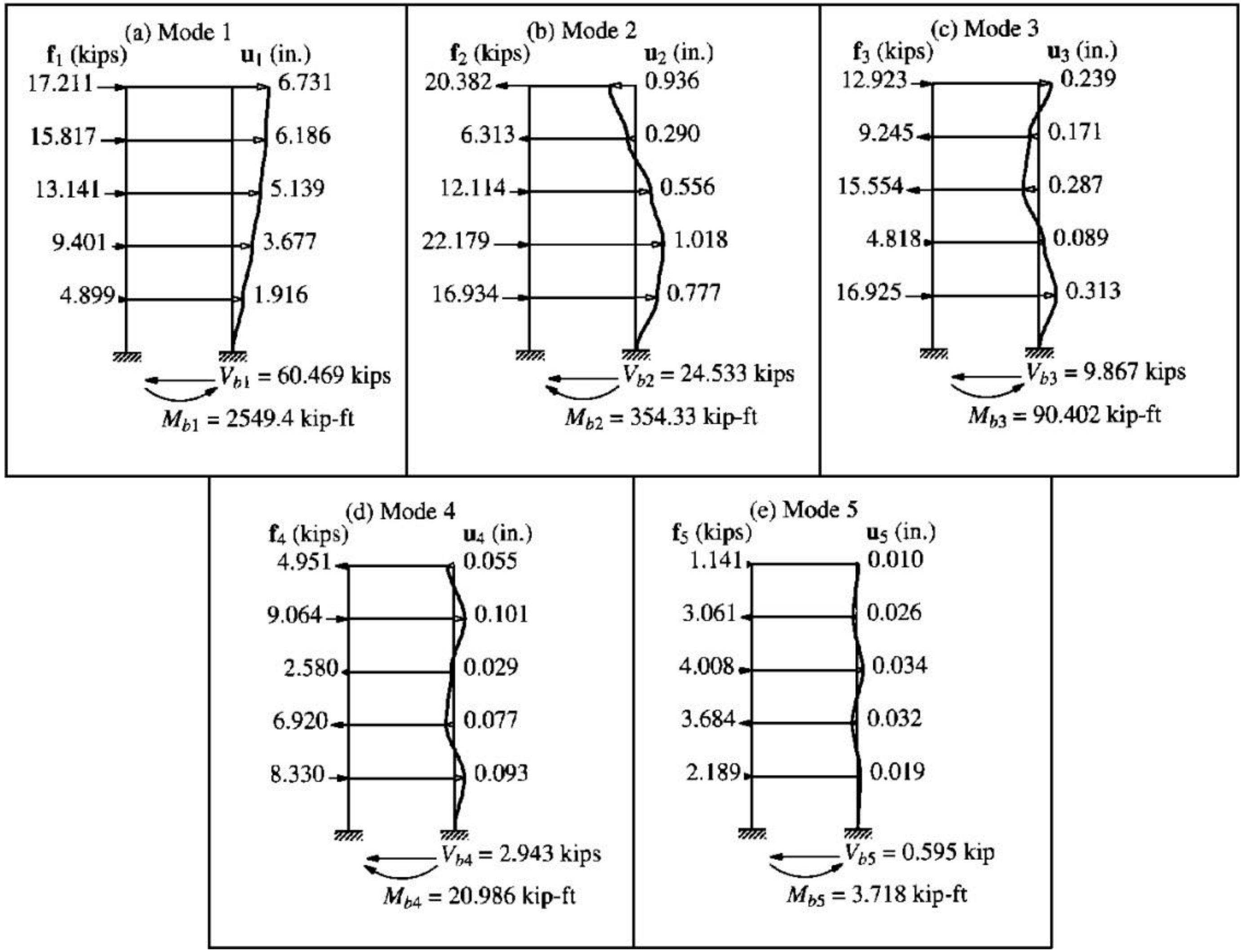
$$\alpha_1 = 0.285,\ \alpha_2 = 0.831,\ \alpha_3 = 1.310,\ \alpha_4 = 1.682, \text{ and } \alpha_5 = 1.919.$$





D_1 = 5.378 in.
D_2 = 2.583 in.
D_3 = 1.505 in.
D_4 = 0.877 in.
D_5 = 0.653 in.
A_3 = 0.8149g
A_4 = 0.7837g
A_2 = 0.5628g
A_1 = 0.1375g
A_5 = 0.7585g
Pseudo-acceleration A, g
Deformation D, in.
Pseudo-velocity V, in./sec
Natural vibration period T_n, sec
T_5
T_4
T_3
T_2
T_1
100
50
20
10
5
2
1
0.5
0.2
0.02
0.05
0.1
0.2
0.5
1
2
5
10
20
50
0.01
0.1
1
10
100
0.001
0.01
0.1
1
10

Modal displacements and modal (equivalent) static forces



PEAK MODAL RESPONSES

Mode	V_b (kips)	V_5 (kips)	M_b (kip-ft)	u_5 (in.)
1	60.469	17.211	2549.4	6.731
2	24.533	−20.382	−354.33	−0.936
3	9.867	12.923	90.402	0.239
4	2.943	−4.951	−20.986	−0.055
5	0.595	1.141	3.718	0.010

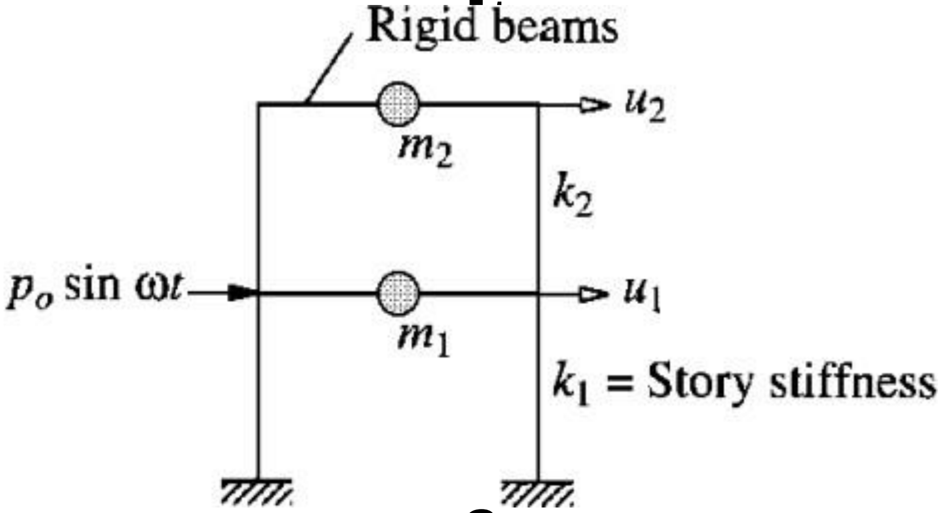
RSA and RHA values of peak response

	V_b (kips)	V_5 (kips)	M_b (kip-ft)	u_5 (in.)
ABSSUM	98.407	56.608	3018.8	7.971
SRSS	66.066	30.074	2575.6	6.800
CQC	66.507	29.338	2572.7	6.793
RHA	73.278	35.217	2593.2	6.847

End of presentation

M

DOF



Consider the 2DOF system subjected to harmonic force as shown

Equations of motion of the system are

e

m

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_o \\ 0 \end{Bmatrix} \sin \omega t$$

u

Above equations are coupled through stiffness matrix

b

.

MDOF sysems b.

Solution of the equation is assumed as $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} \sin \omega t$

Substituting this solution in the equations of motion we get

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\,\omega^2 \end{bmatrix} \begin{Bmatrix} u_{1o} \\ u_{2o} \end{Bmatrix} = \begin{Bmatrix} p_o \\ 0 \end{Bmatrix}$$

We take specific example with
$m_1 = 2m$, $m_2 = m$, $k_1 = 2k$, $k_2 = k$

$\omega_1 = \sqrt{k/2m}$ and $\omega_2 = \sqrt{2k/m}$; Two natural frequencies of vibration

M DO s s t e m s u b .

$$u_{1o} = \frac{p_o(k - m\omega^2)}{2m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \qquad u_1 = u_{1o} \sin \omega t$$

$$u_{2o} = \frac{p_o k}{2m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \qquad u_2 = u_{2o} \sin \omega t$$

Solution for motion of two masses is given above

MDOFsystemsub.

For a MDOF system, governing differential equations are coupled

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$

Solution of these equations, when DOF are large in no, is an extremely difficult task.
Easier solution can be obtained by transforming these equations into a large no of uncoupled equations each to be solved independently.

Decoupling of equations of motion

The displacement vector of any MDOF system can be expanded in Terms of modal contribution

$$U(t) = \sum_{r=1}^{N} \varphi_r q_r(t)$$

Substituting this into equation of motion of MDOF system

$$\sum_{r=1}^{N} \mathbf{m}\,\phi_r \ddot{q}_r(t) + \sum_{r=1}^{N} \mathbf{k}\,\phi_r q_r(t) = \mathbf{p}(t)$$

Premultiplying each term in this equation by ϕ_n^T gives

$$\sum_{r=1}^{N} \phi_n^T\,\mathbf{m}\,\phi_r\,\ddot{q}_r(t) + \sum_{r=1}^{N} \phi_n^T\,\mathbf{k}\,\phi_r\,q_r(t) = \phi_n^T\,\mathbf{p}(t)$$

Decoupling of equations of motion contd

Orthogonality conditions give

$$\phi_n^T \mathbf{k} \phi_r = 0 \qquad \phi_n^T \mathbf{m} \phi_r = 0 \qquad \omega_n \neq \omega_r,$$

The coupled equations are now reduced to a single independent equation for mode n

$$(\phi_n^T \mathbf{m} \phi_n) \ddot{q}_n(t) + (\phi_n^T \mathbf{k} \phi_n) q_n(t) = \phi_n^T \mathbf{p}(t)$$

Or in compact form we write

$$M_n \ddot{q}_n(t) + K_n q_n(t) = P_n(t)$$

$$M_n = \phi_n^T \mathbf{m} \phi_n \qquad K_n = \phi_n^T \mathbf{k} \phi_n \qquad P_n(t) = \phi_n^T \mathbf{p}(t)$$

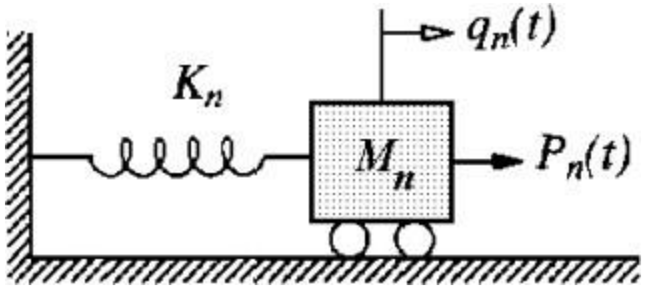
These are generalized mass, stiffness and force for mode n

Decoupling of equations of motion contd

Dividing throughout by M_n we get

$$\ddot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n} \qquad \text{where} \qquad K_n = \omega_n^2 M_n$$

This equation may be interpreted as the equation governing the response q_n (t) of SDOF system shown below with mass M_n, stiffness K_n, and exciting force P_n

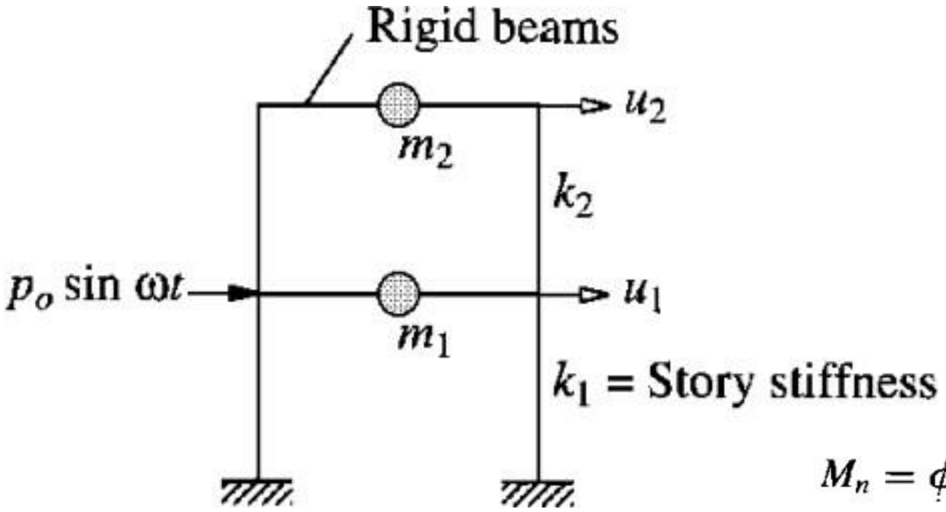


Generalized SDOF system for the n th natural mode

There are N such equations, one for each mode, each independent of other

m_1=2m, m_2=m

K_1=2k, k_2 = k



$$M_n = \phi_n^T \mathbf{m} \phi_n \qquad K_n = \phi_n^T \mathbf{k} \phi_n \qquad P_n(t) = \phi_n^T \mathbf{p}(t)$$

$$\omega_1 = \sqrt{\frac{k}{2m}} \qquad P_2(t) = \phi_2^T \mathbf{p}(t) = \underbrace{-p_o}_{P_{2o}} \sin \omega t$$

$$\phi_1 = \langle \tfrac{1}{2} \quad 1 \rangle^T \qquad \phi_2 = \langle -1 \quad 1 \rangle^T$$

$$M_1 = \frac{3m}{2} \qquad K_1 = \frac{3k}{4}$$

General mass and stiffness mode 1

$$M_2 = 3m \qquad K_2 = 6k$$

General mass and stiffness mode 2

$$P_1(t) = \phi_1^T \mathbf{p}(t) = \underbrace{(p_o/2)} \sin \omega t$$

General force mode 1

$$P_2(t) = \phi_2^T \mathbf{p}(t) = \underbrace{-p_o}_{P_{2o}} \sin \omega t$$

General force mode 2

$$M_n \ddot{q}_n + K_n q_n = P_{no} \sin \omega t$$ Modal equation of motion for mode n

Solution of the equation of motion for n th mode is

$$q_n(t) = \frac{P_{no}}{K_n} C_n \sin \omega t$$ Where $C_n = \frac{1}{1-(\omega/\omega_n)^2}$ Is the DLF for N^{th} mode

$P_{10} = P_0/2$, $K_1 = 3k/4$, Modal force mode 1

$q_1(t) = 2P_0/3k\ C_1 \sin \omega t$ Modal displacement mode 1

$P_{20} = -P_0$, $K_2 = 6k$, Modal force mode 2

$q_2(t) = -P_0/6k\ C_2$ Modal displacement mode 2

Displacements at mass point from modal displacement

$u_1(t) = q_1(t) \times \phi_{11} + q_2(t) \times \phi_{12}$

$= [2P_0/3k \times \frac{1}{2} \times C_1 + P_0/6k \times (-1) \times C_2] \sin \omega t$

$= P_0/6k\,(2C_1 + C_2) \sin \omega t$ displacement of mass point 1

Where $C_1 = \dfrac{1}{1-(\omega/\omega_1)^2}$

And $C_2 = \dfrac{1}{1-(\omega/\omega_2)^2}$

$u_2(t) = \dfrac{p_o}{6k}\,(4C_1 - C_2)\,\sin \omega t$ Displacement of mass point 2

Several dynamic forces on MDOF system with same time variation

Modal expansion of dynamic force **p**(t) = s p(t) where all forces have same time variation and s represents their spatial distribution

We expand the vector s as

$$\mathbf{s} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \, \mathbf{m} \, \phi_r$$

Pre-multiply both sides by ϕ_n^T and use orthogonality principle

We get $$\Gamma_n = \frac{\phi_n^T \mathbf{s}}{M_n}$$

Several dynamic forces on MDOF system with same time variation contd

The contribution of the nth mode to the excitation vector $\mathbf{s}$ is

$$\mathbf{s}_n = \Gamma_n \mathbf{m} \phi_n$$

$$\mathbf{s} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \mathbf{m} \phi_r$$

This can be taken as an expansion of applied force distribution **s** in terms of inertia force distribution s_n in terms of natural modes

If the structure vibrates in n^{th} mode, inertia forces are

$$(\mathbf{f}_I)_n = -\mathbf{m}\ddot{\mathbf{u}}_n(t) = -\mathbf{m}\,\phi_n\,\ddot{q}_n(t)$$

Several dynamic forces on MDOF system with same time variation contd

Thus we conclude from equation $\mathbf{s} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \mathbf{m} \phi_r$

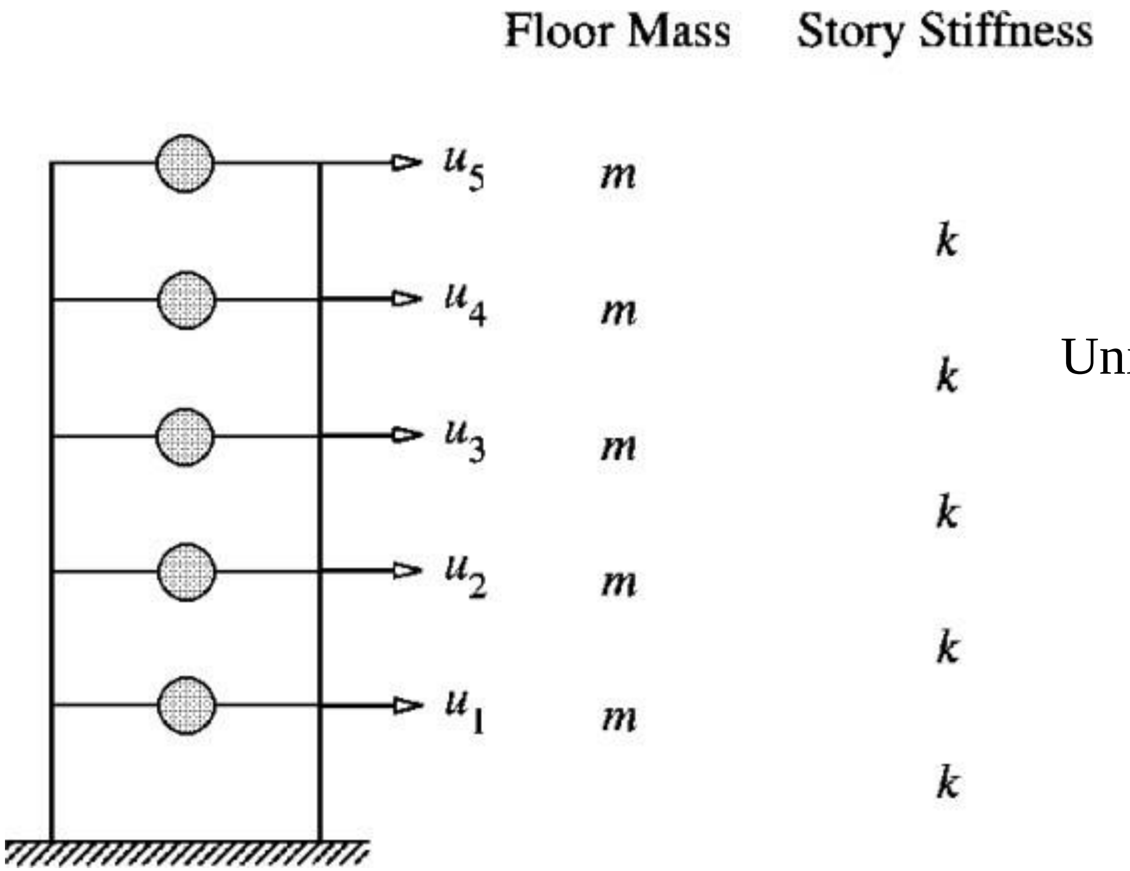
That force $\mathbf{s}_n p(t)$ produces response only in nth mode

Thus force $P_r(t) = \phi_r^T \mathbf{s}_n p(t) = \Gamma_n (\phi_r^T \mathbf{m} \phi_n) p(t)$

Because of orthogonality principle $P_r(t) = 0 \qquad r \neq n$

for $r = n$ is $P_n(t) = \Gamma_n M_n p(t)$

Several dynamic forces on MDOF system with same time variation contd example 1



Uniform 5 storey shear building

Several dynamic forces on MDOF system with same time variation example 1 contd

The mass and stiffness matrices of the structure are

$$\mathbf{m} = m\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \qquad \mathbf{k} = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

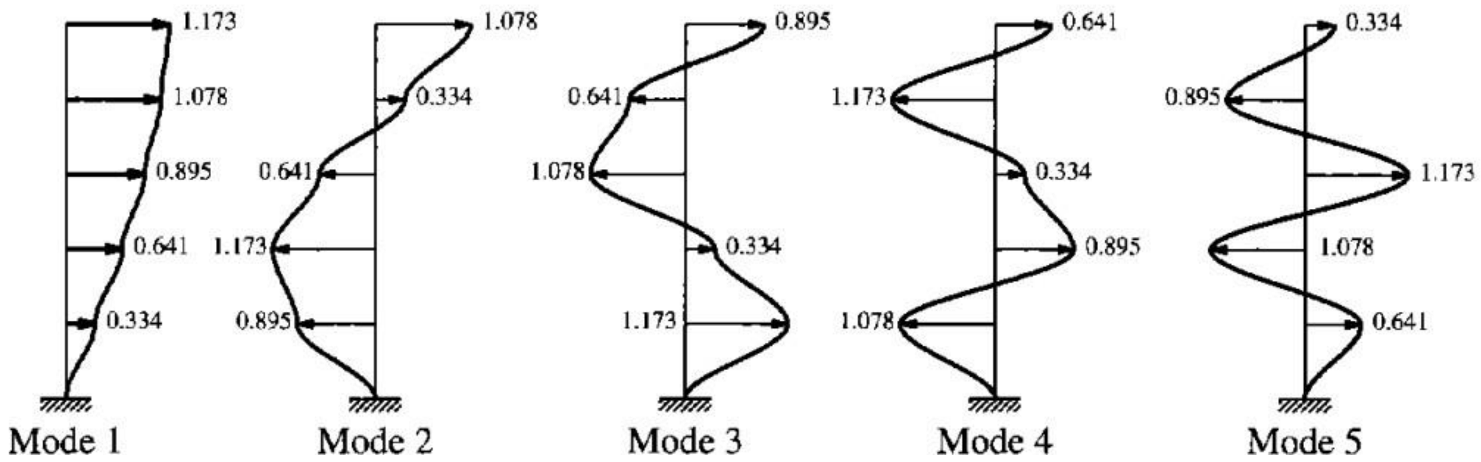
Natural frequencies are $\omega_n = \alpha_n \left(\frac{k}{m}\right)^{1/2}$

$$\alpha_1 = 0.285,\ \alpha_2 = 0.831,\ \alpha_3 = 1.310, \alpha_4 = 1.682,\ \alpha_5 = 1.919$$

Several dynamic forces on MDOF system with same time variation example 1 contd

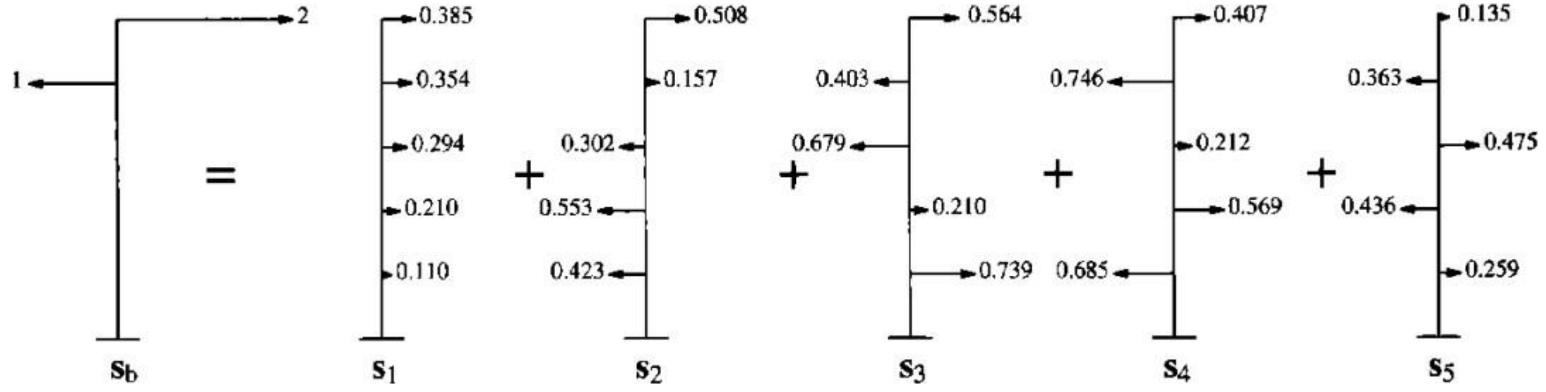
Natural modes of vibration of uniform five-story shear building.

$$\phi_1 = \begin{Bmatrix} 0.334 \\ 0.641 \\ 0.895 \\ 1.078 \\ 1.173 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} -0.895 \\ -1.173 \\ -0.641 \\ 0.334 \\ 1.078 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 1.173 \\ 0.334 \\ -1.078 \\ -0.641 \\ 0.895 \end{Bmatrix} \quad \phi_4 = \begin{Bmatrix} -1.078 \\ 0.895 \\ 0.334 \\ -1.173 \\ 0.641 \end{Bmatrix} \quad \phi_5 = \begin{Bmatrix} 0.641 \\ -1.078 \\ 1.173 \\ -0.895 \\ 0.334 \end{Bmatrix}$$



Several dynamic forces on MDOF system with same time variation example 1 contd

Forces and their modal representation



Modal equation

In Modal equation of motion for n^{th} mode

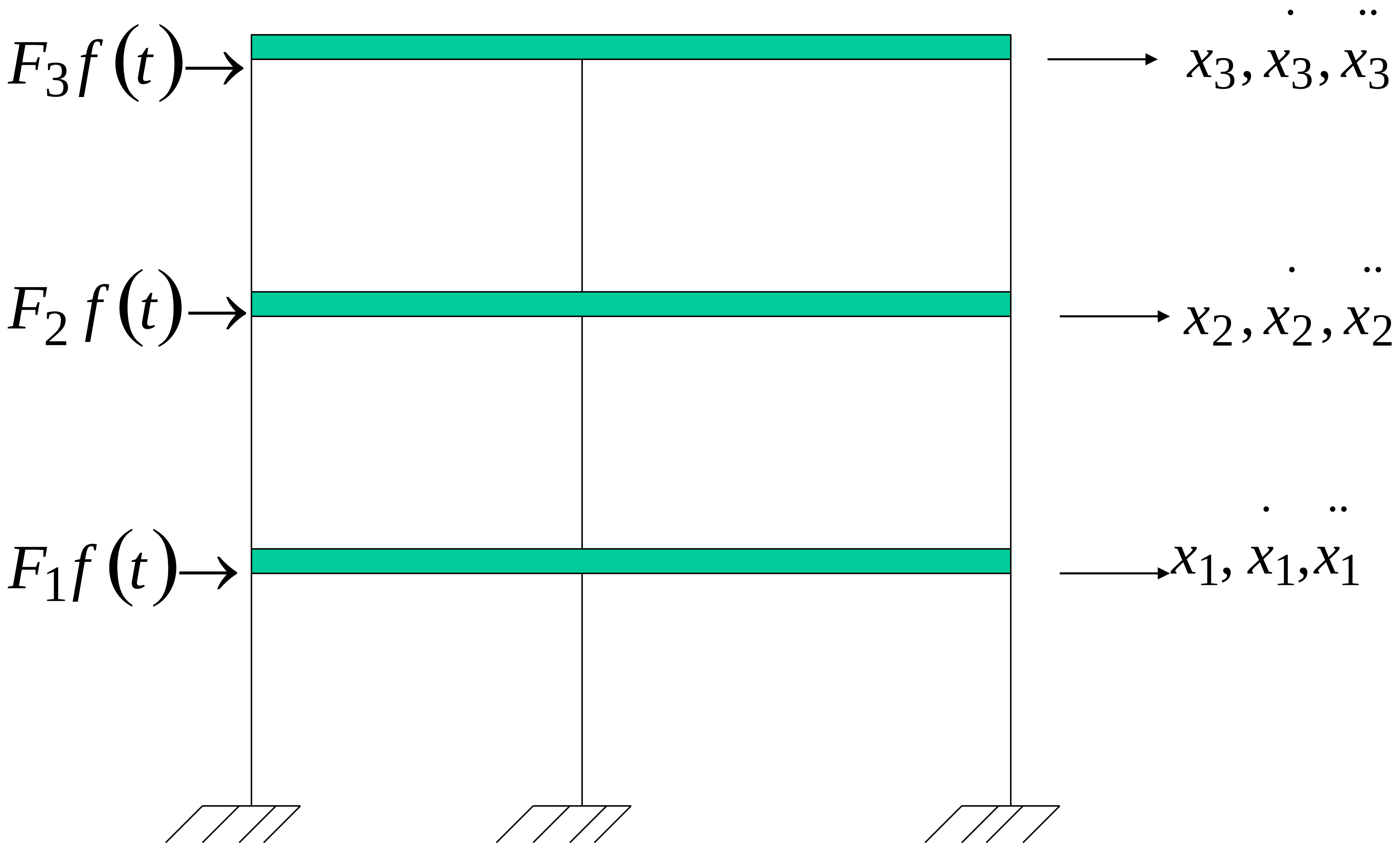
$$\ddot{q}_n + 2\zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n}$$

Substitute for modal force $P_n(t) = \Gamma_n M_n p(t)$ we get

$$\ddot{q}_n + 2\zeta_n \omega_n \dot{q}_n + \omega_n^2 q_n = \Gamma_n p(t)$$

The factor Γ_n Is called modal participation factor

is a measure of the degree to which the *n*th mode participates in the response.


$F_3 f(t)$
$F_2 f(t)$
$F_1 f(t)$
$x_3, \dot{x}_3, \ddot{x}_3$
$x_2, \dot{x}_2, \ddot{x}_2$
$x_1, \dot{x}_1, \ddot{x}_1$

When the vibratory motion is set in following equations of motion can be written in directions of different DOF

$$K_{11}x_1 + K_{12}x_2 + K_{13}x_3 + ...+ K_{1n}x_n + M_1 \ddot{x}_1 = F_1 f(t)$$

$$K_{21}x_1 + K_{22}x_2 + K_{23}x_3 + ...+ K_{2n}x_n + M_2 \ddot{x}_2 = F_2 f(t)$$

$$K_{31}x_1 + K_{32}x_2 + K_{33}x_3 + ...+ K_{3n}x_n + M_3 \ddot{x}_1 = F_3 f(t)$$

In compact form the above equations can be written as

$$[K]\{x\} + [M]\{\ddot{x}\} = f(t)\{F\}$$

The shape that a structure takesat any instant can be split into it's modal components. Thus we can express

$$\{x\} = Y\phi = Y_1\phi_1 + Y_2\phi_2 + Y_3\phi_3 + Y_4\phi_4 + ... + Y_n\phi_n$$

Where Y are the so called weightages to mode shape

and ϕ are the mode shapes for natural vibrations.

Differentiating both sides weget

$$\{\ddot{x}\} = \ddot{Y}\phi = \ddot{Y}_1\phi_1 + \ddot{Y}_2\phi_2 + \ddot{Y}_3\phi_3 + \ddot{Y}_4\phi_4 + ... + \ddot{Y}_n\phi_n$$

Substituting the expressions for x and $\ddot{x}$ we get

$$[K]\{\phi\}Y + [M]\{\phi\}\ddot{Y} = f(t)\{F\}$$

This is a set of n simultaneous ordinary differential equations with constant coefficients.

Pre-multiplying both sides by n^{th} mode shape ϕ_n and using orthogonality principle we reduce the equations as follows

$$\{\phi_n\}^T [K]\{\phi\}Y + \{\phi_n\}^T [M]\{\phi\}\ddot{Y} = f(t)\{\phi_n\}^T \{F\}$$

$$\{\phi_n\}^T [K]\{\phi_n\}Y_n + \{\phi_n\}^T [M]\{\phi_n\}\ddot{Y}_n = f(t)\{\phi_n\}^T \{F\}$$

$$[K]_{eq}^{n} Y_n + [M]_{eq}^{n} \ddot{Y}_n = f(t)\{F\}_{eq}^{n}$$

Where

$$[K]_{eq}^{n} = \{\phi_n\}^T [K] \{\phi_n\} Y_n = \text{Equivalent stiffness for } n^{th} \text{ mode}$$

$$[M]_{eq}^{n} = \{\phi_n\}^T [M] \{\phi_n\} = \text{Equivalent mass for } n^{th} \text{ mode}$$

$$\{F\}_{eq}^{n} = \{\phi_n\}^T \{F\} = \text{Equivalent force for } n^{th} \text{ mode}$$

This equation is similar to governing equation of motion for SDOF system subjected to specified dynamic force.

$$M\ddot{x} + Kx = f(t)F$$

Calculations

- SDOF

$$Y_{dynamic} = Y_{Static} * DLF$$

$$Y_{static} = \frac{F_o}{K}$$

DLF --- function of T and f(t)

- MDOF

$$Y_{dynamic}^{(n)} = Y_{Static}^{(n)} * DLF^{(n)}$$

$$Y_{static}^{(n)} = \frac{F_{eq}^{(n)}}{K_{eq}^{n}} = \frac{F_{eq}^{(n)}}{M_{eq}^{n} \omega^{2}}$$

DLF^{n} --- function of T_n and f(t)

Further calculations for MDOF

Modal dynamic deflection = Modal static deflection * Modal DLF

Modal static deflection = Modal Equivalent force/Modal stiffness

$$\text{Modal Equivalent force} = \phi_{\mathrm{n}}^{\mathrm{T}} * F$$

$$\text{Modal Equivalent stiffness} = \phi_{\mathrm{n}}^{\mathrm{T}} * K * \phi_n = \omega_n^2 \phi_{\mathrm{n}}^{\mathrm{T}} * M * \phi_n$$

$$\text{Deflection of a mass point} = \phi_{\mathrm{n}}^{\mathrm{T}} * \text{Modal dynamic deflection}$$

Repeat the calculations per mode

Add modal Dynamic deflections at each mass point as

1. Absolute sum
2. SRSS
3. CQC type addition